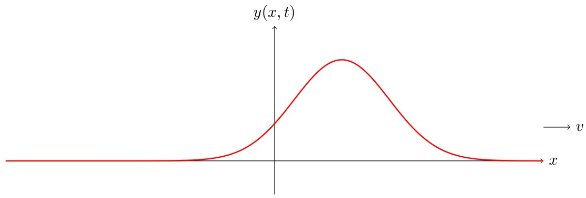
practice midterm 2 walkthrough: video links in announcement.

Problem 2 (20 points)



Suppose we have a very long string with linear mass density μ. A Gaussian-shaped wave propagates to the right with velocity v, described as follows:

$$y(x,t) = Ae^{-(x-vt)^2/2\sigma^2}$$

a) Verify that the Gaussian function satisfies the 1-dimensional wave equation. (5 points)
b) Solve for the energy of the entire Gaussian wave. You may use the following integral:

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

(10 points)
c) Solve for the magnitude of the maximal transverse velocity of the wave. (5 points)

a) $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

$$\frac{\partial y}{\partial x} = Ae^{-\frac{(x-vt)^2}{2\sigma^2}} \cdot \frac{\partial}{\partial x}\left(-\frac{(x-vt)^2}{2\sigma^2}\right) = y \cdot \left[-\frac{1}{2\sigma^2} \cdot 2(x-vt) \cdot 1\right]$$

$$= \underbrace{-\frac{x-vt}{\sigma^2}}_{v} \underbrace{Ae^{-\frac{(x-vt)^2}{2\sigma^2}}}_{u}$$

$(uv)' = u'v + uv'$

$$\frac{\partial^2 y}{\partial x^2} = \left[\frac{\partial}{\partial x}\left(Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\right)\right]\left(-\frac{x-vt}{\sigma^2}\right) + Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\left[\frac{\partial}{\partial x}\left(-\frac{x-vt}{\sigma^2}\right)\right]$$

$$= \frac{(x-vt)^2}{\sigma^4} Ae^{-\frac{(x-vt)^2}{2\sigma^2}} + Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\left(-\frac{1}{\sigma^2}\right)$$

$$= \boxed{Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\left(\frac{(x-vt)^2}{\sigma^4} - \frac{1}{\sigma^2}\right)}$$

$\times v^2$

$$\frac{\partial y}{\partial t} = \underbrace{\frac{v(x-vt)}{\sigma^2}}_{v} \underbrace{Ae^{-\frac{(x-vt)^2}{2\sigma^2}}}_{u}$$

$$\frac{\partial^2 y}{\partial t^2} = \left[\frac{\partial}{\partial t} Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\right]\frac{v(x-vt)}{\sigma^2} + Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\left[\frac{\partial}{\partial t}\frac{v(x-vt)}{\sigma^2}\right]$$

$$= Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\left(v^2\frac{(x-vt)^2}{\sigma^4}\right) + Ae^{-\frac{(x-vt)^2}{2\sigma^2}}\left(-\frac{v^2}{\sigma^2}\right) = \boxed{Ae^{-\frac{(x-vt)^2}{2\sigma^2}} \cdot v^2\left(\frac{(x-vt)^2}{\sigma^4} - \frac{1}{\sigma^2}\right)}$$

b) $E = \frac{1}{2}\mu \int_{-\infty}^{\infty} \left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right] dx$

$\left(\frac{\partial y}{\partial t}\right)^2 = \frac{v^2 (x-vt)^2}{\sigma^4} y^2$, $\left(\frac{\partial y}{\partial x}\right)^2 = \frac{(x-vt)^2}{\sigma^4} y^2 \Longrightarrow \left(\frac{\partial y}{\partial t}\right)^2 = v^2\left(\frac{\partial y}{\partial x}\right)^2$

$\Longrightarrow E = \mu v^2 \int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial x}\right)^2 dx = \mu v^2 \int_{-\infty}^{\infty} \frac{(x-vt)^2}{\sigma^4} A^2 e^{-\frac{(x-vt)^2}{\sigma^2}} dx$

$u = \frac{x-vt}{\sigma} \Longrightarrow du = \frac{dx}{\sigma} \longrightarrow dx = \sigma du$

$E = \mu v^2 \frac{A^2}{\sigma^4} \int_{-\infty}^{\infty} (\sigma u)^2 e^{-u^2} \sigma du = \mu v^2 \frac{A^2}{\sigma} \int_{-\infty}^{\infty} u^2 e^{-u^2} du = \boxed{\frac{\mu v^2 A^2 \sqrt{\pi}}{2\sigma}}$

c) $\left(\frac{(x-vt)^2}{\sigma^4} - \frac{1}{\sigma^2}\right) = 0$

$\frac{(x-vt)^2}{\sigma^4} = \frac{1}{\sigma^2}$

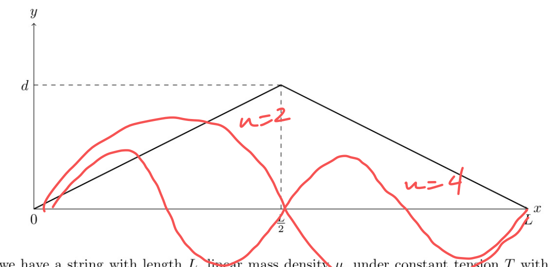
$(x-vt)^2 = \sigma^2$

$\rightarrow |x-vt| = \sigma \Longrightarrow \left|\frac{\partial y}{\partial t}\right|_{max} = \frac{v(x-vt)}{\sigma^2} A e^{-\frac{(x-vt)^2}{2\sigma^2}}$

$= \frac{v\cancel{\sigma}}{\sigma^{\cancel{2}}} \cdot A e^{-\frac{\cancel{\sigma^2}}{2\cancel{\sigma^2}}}$

$= \boxed{\frac{vA}{\sigma} e^{-1/2}}$

Problem 3 (25 points)



Suppose we have a string with length L, linear mass density μ, under constant tension T with its ends at $x = 0$ and $x = L$ fixed. The midpoint of the string is displaced by a distance d and released at time $t = 0$.

a) $f(x)$ models the shape of the string at $t = 0$. Solve for $f(x)$. (5 points)

b) Find the amplitude of the n-th normal mode excited in the string. (10 points)

c) Using your answer from part **b**, for which values of n is the amplitude of the n-th normal mode zero? (5 points)

d) Solve for the energy of the n-th normal mode. (5 points)

a)
$$f(x) = \begin{cases} \frac{2d}{L}x, & 0 \le x \le L/2 \\ 2d - \frac{2d}{L}x, & L/2 \le x \le L. \end{cases}$$

b)
$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$
$$= \frac{2}{L}\left(\int_0^{L/2}\frac{2d}{L}x\sin(\quad)dx + \int_{L/2}^{L}2d\sin(\quad)dx - \int_{L/2}^{L}\frac{2d}{L}x\sin(\quad)dx\right].$$

1st term
$$= \frac{4d}{L^2}\int_0^{L/2}x\sin\left(\frac{n\pi}{L}x\right)dx = uv\Big|_0^{L/2} - \int_0^{L/2}v\,du$$

$u = x \qquad dv = \sin\left(\frac{n\pi}{L}x\right)dx$

$du = dx \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)$

$$= -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2} + \int_0^{L/2}\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)dx$$
$$= -\frac{L}{2}\cdot\frac{L}{n\pi}\cos\left(\frac{n\pi\cancel{L}}{2\cancel{L}}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{L}x\right)\Big|_0^{L/2}$$
$$= \left[-\frac{\cancel{L^2}}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{\cancel{L^2}}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]\frac{4d}{\cancel{L^2}}$$

2nd term
$$= \frac{4d}{L}\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)dx = \frac{4d}{\cancel{L}}\left[-\frac{\cancel{L}}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{L/2}^{L}$$
$$= -\frac{4d}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

3rd term :

$$= \frac{-4d}{L^2} \int_{L/2}^{L} x \sin\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{-4d}{L^2} \left\{ \left[-x \cdot \frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right)\right]_{L/2}^{L} + \int_{L/2}^{L} \frac{L}{n\pi} \cos\left(\frac{n\pi}{L}x\right) dx \right\}$$

$$= \frac{-4d}{L^2} \left\{ \left[-L \frac{L}{n\pi} \cos(n\pi) + \frac{L}{2} \frac{L}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{L}x\right) \right|_{L/2}^{L} \right\}$$

$$= \frac{-4d}{L^2} \left\{ \left[-\frac{L^2}{n\pi} \cos(n\pi) + \frac{L^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right)\right] + \left[\frac{L^2}{n^2\pi^2} \sin(n\pi) - \frac{L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)\right] \right\}$$

collect terms

$\sum \cos(n\pi)$ terms $= 0$

$\sum \cos(n\pi/2)$ " $= 0$

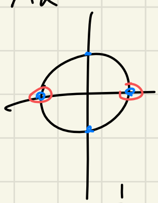
$\frac{4d}{n^2\pi^2} \sin(n\pi) = 0$

$$2 \cdot \frac{4d}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{8d}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \boxed{A_n = \frac{8d}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)}$$

c) $A_n = 0$? when $n = 2, 4, 6, \ldots$

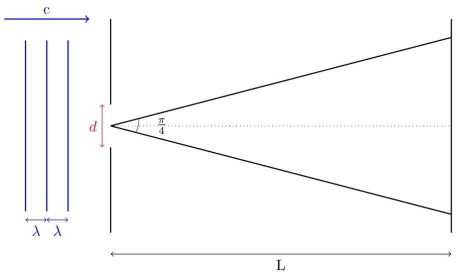


d) $E_n = \frac{1}{4} \mu L A_n^2 \omega_n^2$, $\quad \omega_n = \frac{n\pi v}{L} \leftarrow \quad v = \sqrt{\frac{T}{\mu}}$

$$= \frac{1}{4} \mu L \left(\frac{8d}{n^2\pi^2}\right)^2 \sin^2\left(\frac{n\pi}{2}\right) \frac{n^2\pi^2 v^2}{L^2} \leftarrow \frac{T}{\mu}$$

$$= \frac{16 d^2 T}{n^2 \pi^2 L} \sin\left(\frac{n\pi}{2}\right)$$

Problem 4 (15 points)

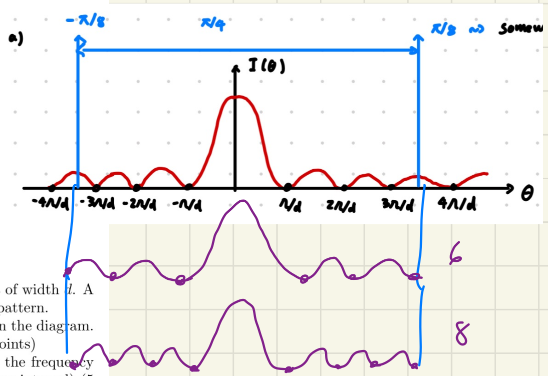


Suppose we have monochromatic light of wavelength λ and frequency ν passing through a slit of width d. A screen is set at a distance $L >> d$ to the right of the slit where we can observe a diffraction pattern.

a) In the diffraction pattern, we observe 6 minima within an angular interval $\frac{\pi}{4}$ as indicated in the diagram. Determine the minimum and maximum values of λ to produce this effect on the screen. (5 points)

b) Suppose λ is the maximal value you found in part **a**. By what factor should we increase the frequency ν to **just** observe 8 minima within the angular interval? (HINT: this is a single value, not an interval) (5 points)

c) Suppose the slit width d is halved. Describe the qualitative changes to the diffraction pattern. (5 points)



a) $d\sin\theta = n\lambda$

$\theta = \frac{n\lambda}{d}$, $n = 1, 2, 3, \ldots$

see practice midterm 2 sol'ns for calc.

$$\frac{3\lambda}{d} < \frac{\pi}{8} < \frac{4\lambda}{d} \Rightarrow \boxed{\frac{\pi d}{32} < \lambda < \frac{\pi d}{24}}$$

b) $\lambda = \frac{\pi d}{24}$, $\lambda' = \frac{\pi d}{32} \Rightarrow \frac{\lambda'}{\lambda} = \frac{24}{32} = \frac{3}{4}$

$$\frac{\nu'}{\nu} = \frac{4}{3} \Rightarrow \boxed{\nu' = \frac{4}{3}\nu}$$

c) $\theta = \frac{\lambda}{d}$

$\theta' = \frac{\lambda}{d'} = \frac{2\lambda}{d}$

$d' = d/2$